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The statistical error in X-ray absorption measurements

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With 5 figures in the text

ABSTRACT

The statistical error in the determination of X-ray absorption coefficients is studied for the case of counter registration. Optimum conditions are found for the absorber thickness and for the time distribution between measuring I_0 and I. The influence of background radiation is discussed, and a table giving the optimum conditions for different backgrounds is appended.

Introduction

X-ray absorption coefficients are generally determined by observing the radiation intensity with and without an absorber placed in the path of the ray. If the thickness d of the absorber is known, the linear absorption coefficient μ can be found from

$$I = I_0 e^{-\mu d}, \tag{1}$$

where I_0 and I are the incident and the transmitted intensities respectively.

When the intensities are registered by means of a photon counter, statistical errors due to the random time distribution of the photons are introduced. As is well known the standard deviation when counting N pulses is $\sqrt{N}$, and consequently the relative error decreases if the counting time (and therefore N) increases.

However, even with a constant total counting time (= the time necessary for an absorption coefficient measurement, i.e. the sum of the times spent on measuring I_0 and I separately), there are two parameters which can be varied in order to minimize the statistical error in μ, namely the absorber thickness d and the distribution of the total available time between measuring I_0 and I.

As to the choice of d several expressions for "optimum thickness" have been deduced. Sandström [1] and Kurylenko [2] arrive at theirs from certain requirements for the intensity contrasts in the transmitted radiation. Thus their values of d_{opt} are significant only when this radiation alone is recorded, e.g. by means of a photographic method. Rose and Shapiro [3] have carried out error caculations for a counter determination of absorption coefficients in general on lines similar to those presented in this paper. They assume that the background radiation is constant, i.e. not affected by the absorber. However, as will be shown later, this is normally not the case in X-ray spectroscopy. Moreover the varying degrees of background radiation absorption strongly influence the optimum thickness of the absorber, according to Fig. 1. A paper by Parratt [4] must be mentioned in this connection. He finds that the shape of an absorption spectrum varies with absorber thickness, owing to the spectral window. The influence of this "thickness effect" decreases with decreasing absorber thickness, and from this point of view the absorber should be as thin as is practically possible. In general the statistical error will then increase, and the final choice of absorber thickness will have to be a compromise between statistical precision and physical accuracy.

Error calculation

a. *Background radiation not negligible*

In a practical case the linear absorption coefficient μ is calculated from the equation

$$n-\nu=(n_0-\nu_0)\,e^{-\mu d}, \tag{2}$$

where n and n_0 are counting rates with and without the absorber, and ν, ν_0 are the corresponding background radiation rates. Solving (2) for μ gives

$$\mu=\frac{1}{d}\left[\log\,(n_0-\nu_0)-\log\,(n-\nu)\right], \tag{3}$$

which is the starting point for our error calculations.

As to ν_0 and ν some assumptions must be made in order to prevent the calculations from becoming too complicated.

The background radiation can be said to consist of two parts, one reaching the detector from all directions in space, and the other entering the counter along the same direction as the coherent radiation from the crystal. This part is mainly due to fluorescent radiation from the crystal holder and the crystal itself and has a comparatively high intensity, while the former part, derived from such sources as cosmic radiation and stray X-radiation in general, is usually considerably less intense. In fact, if the counter is energy-sensitive and used in combination with a channel discriminator, it can normally be entirely neglected. For the sake of the error calculation we can thus assume that all background radiation present travels along the direction of the incident coherent radiation. In consequence it, too, undergoes an absorption in the foil, and we can define an average absorption coefficient μ_ν for the background:

$$\nu=\nu_0\,e^{-\mu_\nu d}. \tag{4}$$

ν_0, ν and μ_ν will on the whole be independent of the wavelength setting of the spectrograph, at least within a limited region, e.g. around an absorption edge.

One more assumption concerning background radiation will be made, namely that the statistical error in ν_0 and ν is negligible compared to the error in n_0 and n. That this is justified appears from the practical method of determining ν_0 and ν described later.

In Eq. (3) we therefore have only two statistical variables, namely n_0 and n, and differentiation leads to

$$d\mu=\frac{1}{d}\left(\frac{dn_0}{n_0-\nu_0}-\frac{dn}{n-\nu}\right). \tag{5}$$

If N_0 is the total number of pulses counted without the absorber during time t_0 and N the number counted with the absorber during time t we obtain

$$n_0 = \frac{N_0}{t_0}, \quad n = \frac{N}{t} \tag{6}$$

and

$$d n_0 = \frac{\sqrt{N_0}}{t_0}, \quad d n = \frac{\sqrt{N}}{t}. \tag{7}$$

The standard deviation σ_μ is then

$$\sigma_\mu = \frac{1}{d}\left(\frac{N_0}{t_0^2\,(n_0-\nu_0)^2} + \frac{N}{t^2\,(n-\nu)^2}\right)^{1/2}. \tag{8}$$

With the aid of Eqs. (2), (4) and (6) and introducing a total counting time $T = t_0 + t$ this expression is transformed into

$$\sigma_\mu = \frac{1}{d\,(n_0-\nu_0)}\left[\frac{(T-t_0)\,n_0 + \nu_0 t_0\, e^{d(2\mu-\mu_\nu)} + t_0\,(n_0-\nu_0)\,e^{\mu d}}{t\,(T-t_0)}\right]^{1/2}. \tag{9}$$

Here the variables are d and t_0, and we want to find their optimum values, making σ_μ a minimum. For the sake of simplicity we minimize the variance σ_μ^2, and the conditions are:

$$\frac{\partial}{\partial d}\,\sigma_\mu^2 = 0; \tag{10 a}$$

$$\frac{\partial}{\partial t_0}\,\sigma_\mu^2 = 0. \tag{10 b}$$

This leads to the equation system

$$2y = \alpha\, e^{x(2-\beta)}\,[x(2-\beta)-2] + e^{x}\,(1-\alpha)\,(x-2); \tag{11 a}$$

$$y^2 = \alpha\, e^{x(2-\beta)} + e^{x}\,(1-\alpha), \tag{11 b}$$

where

$$\begin{cases} x = \mu \cdot d, & y = \dfrac{t}{t_0}, \\ \alpha = \dfrac{\nu_0}{n_0}, & \beta = \dfrac{\mu_\nu}{\mu}. \end{cases} \tag{12}$$

The variables d and t_0 appear only in the combinations given by x and y above, and thus the latter can be chosen as new variables. For known values of the parameters α (the relative background intensity) and β (the relative background absorption) x_{opt} and y_{opt} can be determined from the system (11). The optimum thickness then follows from

$$d_{\text{opt}} = \frac{x_{\text{opt}}}{\mu}. \tag{13}$$

A way of finding α and β in a practical case will be shown in the final section. A table giving a pair of optimum values $(x_{\text{opt}}, y_{\text{opt}})$ for different parameters α, β is appended. This has been calculated from system (11) with the aid of BESK, the electronic computer in Stockholm. The values given for $\beta = 0$

(no background absorption) correspond to the special case treated by Rose and Shapiro [3].

According to Eq. (8) the relative standard deviation is

$$\frac{\sigma_\mu}{\mu}=\frac{1}{\mu d}\left(\frac{N_0}{t_0^2\,(n_0-\nu_0)^2}+\frac{N}{t^2\,(n-\nu)^2}\right)^{1/2},$$

which can be transformed into

$$\frac{\sigma_\mu}{\mu}=\frac{1}{\sqrt{N_0}}\cdot\frac{1}{x\,(1-\alpha)}\left(1+\frac{\alpha\, e^{x\,(2-\beta)}+(1-\alpha)\,e^x}{y}\right)^{1/2} \tag{14}$$

with the designation according to (12).

If x and y are at their optimum values this expression is simplified to

$$\frac{\sigma_\mu}{\mu}=\frac{1}{\sqrt{N_0}}\cdot\frac{1}{x_{\text{opt}}\,(1-\alpha)}\,(1+y_{\text{opt}})^{1/2} \tag{15}$$

as in this case $\alpha\, e^{x\,(2-\beta)}+(1-\alpha)\,e^x=y^2$ according to Eq. (11 b).

b. *Background radiation negligible*

A negligible background corresponds to $\alpha=0$ in system (11), which is then considerably simplified:

$$\left\{\begin{aligned} 2\,y&=e^x\,(x-2); && \text{(16 a)}\\ y^2&=e^x, && \text{(16 b)}\end{aligned}\right.$$

with the approximate solution $x_{\text{opt}}=2.56$, $y_{\text{opt}}=3.59$. The optimum conditions are thus

$$\left\{\begin{aligned} d_{\text{opt}}&=\frac{2.56}{\mu};\\ \left(\frac{t}{t_0}\right)_{\text{opt}}&=3.59.\end{aligned}\right. \tag{17}$$

The relative standard deviation here is

$$\frac{\sigma_\mu}{\mu}=\frac{1}{\sqrt{N_0}}\cdot\frac{1}{x}\left(1+\frac{e^x}{y}\right)^{1/2}, \tag{18}$$

which for optimum values of x and y takes the form

$$\frac{\sigma_\mu}{\mu}=\frac{0.84}{\sqrt{N_0}}. \tag{19}$$

This expression represents the smallest error possible when determining absorption coefficients. If a background is present, or if d or $\frac{t}{t_0}$ deviate from their optimum values, the error will increase.

Discussion

A study of the table appended reveals the influence of the background on x_{opt} and y_{opt}. If the very highest combinations of α and β are excluded, the time distribution $\frac{t}{t_0}$ varies but slightly with background conditions. According to Eq. (14) or (18) the relative standard deviation in its turn varies slowly with y in the region considered here. Thus the choice of $\frac{t}{t_0}$ is not critical, and can in most cases be taken *a priori* anywhere in the neighbourhood of $\frac{t}{t_0} = 3.50$.

It should be noted, however, that the above discussion only applies if x has its optimum value. Otherwise $\left(\frac{t}{t_0}\right)_{\text{opt}}$ can change considerably. y_{opt} is then determined solely by Eq. (11 b), but owing to the insensitivity of y to background, Eq. (16 b) is equally accurate for most cases. This gives

$$y_{\text{opt}} = e^{\frac{\mu d}{2}} \tag{20}$$

for any value of d and μ. For example, if $x = \frac{1}{2}\, x_{\text{opt}}$, y_{opt} decreases to 2.2. This variation of y_{opt} has to be remembered when an absorber is used over a region with great variations in μ, a case more fully discussed below.

In order to illustrate the variation of x_{opt} with background conditions, Figs. 1 and 2 have been plotted using values from the table.

Fig. 1 gives x_{opt} as a function of β, the relative background absorption, for three different values of the relative background intensity. When the background is absorbed to a lesser extent than the coherent radiation ($\beta < 1$), d_{opt} is smaller than in the special case $\alpha = 0$; when the background absorption is greater ($\beta > 1$), d_{opt} is greater. This effect increases with increasing background intensity.

Fig. 2 illustrates x_{opt} as a function of α for three different values of β. The most interesting feature of this curve is the sharp decrease of x_{opt} at small values of α, when the relative background absorption is low. Furthermore it is evident that the case $\beta = 1$ is equivalent to $\alpha = 0$, i.e. non-existent background. This is explained by the fact that if $\mu_\nu = \mu$ the background in absorption connections is indistinguishable from the coherent radiation and can thus be treated as a part of the latter.

So far we have only considered the case where an absorption measurement is made at one definite X-ray energy (or in a region where $\mu = \text{constant}$). The normal case, however, is that a variation of μ with energy is studied, e.g. at an absorption edge. According to the relation (13) a different absorber thickness is then required for each μ. In practice such an arrangement is naturally next to impossible, and fortunately it turns out to be unnecessary. Let us study the changes of the relative standard deviation when d deviates from d_{opt}. Eq. (14) states this variation if y is held constant. However, for the sake of simplicity we choose Eq. (18) instead, which is valied for the special case $\alpha = 0$. Fig. 3 shows this function, when $y = 3.59$ and $N_0 = 10^4$. The general case, Eq. (14), gives the same curve but with different absolute values, depending on α and β.

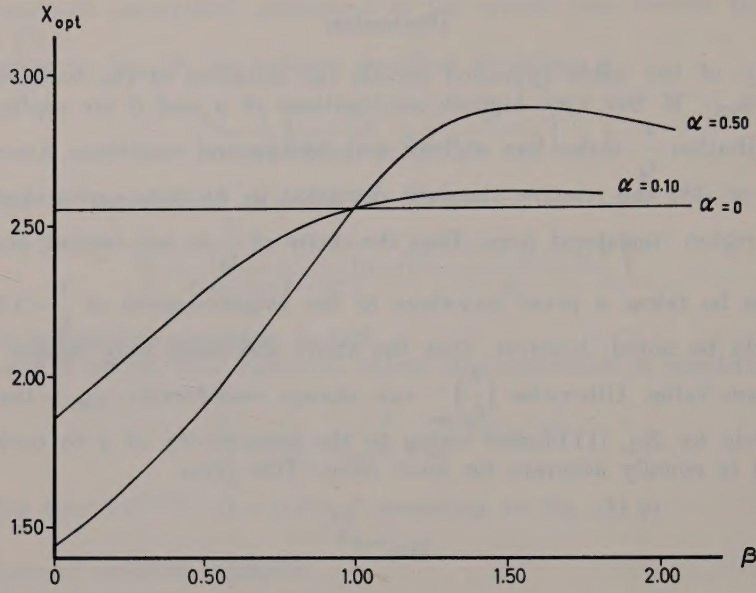

Fig. 1. The optimum thickness d_{opt} as a function of the relative background absorption β for three different values of the relative background α.

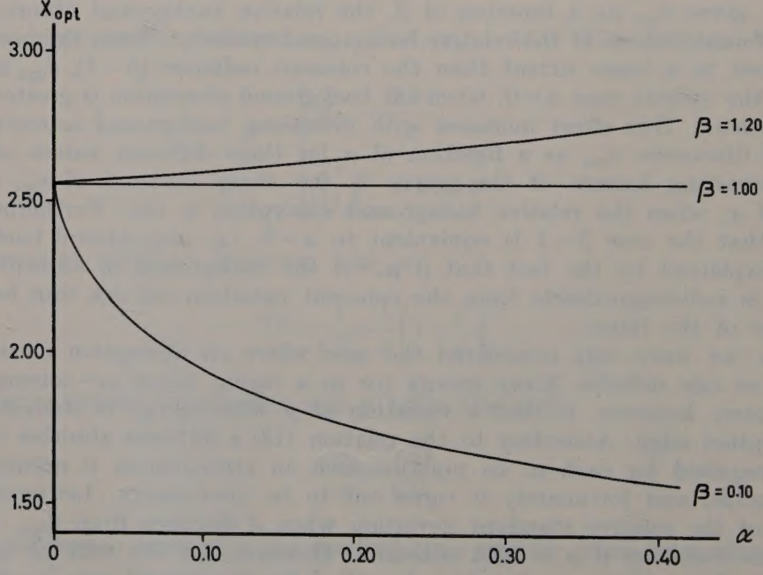

Fig. 2. d_{opt} as a function of α for three different values of β.

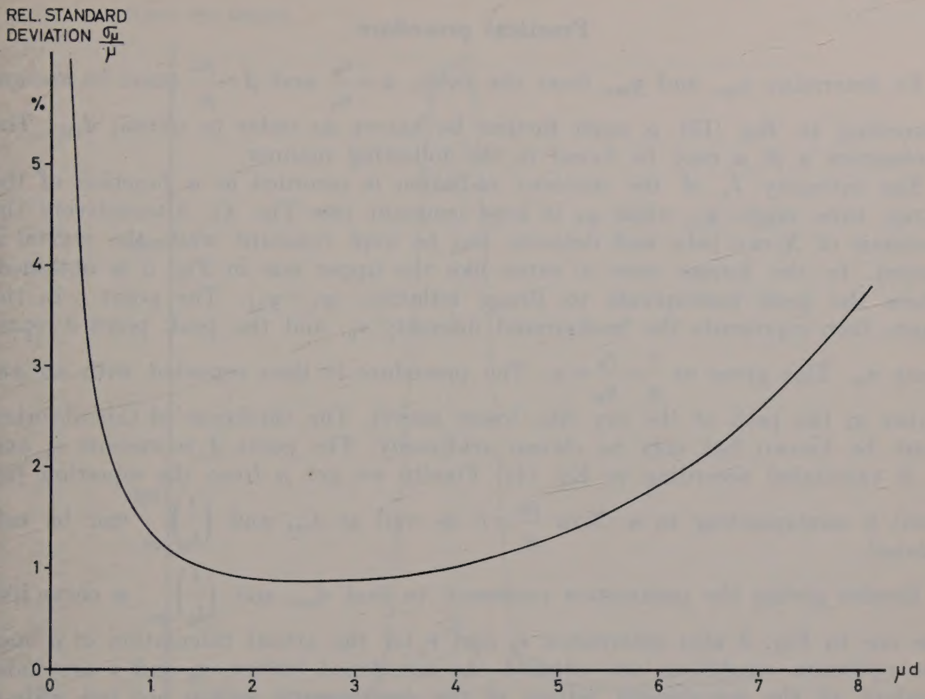

Fig. 3. The relative standard deviation σ_μ/μ as a function of the product μd, according to Eq. (18). $N_0 = 10^4$ and $y = 3.59$.

The characteristic feature of the curve is the broad and flat minimum, which at once indicates that d_{opt} is not a critical quantity. In fact, in this special case any μd-value between 1 and 5 is equally acceptable, giving errors slightly above 1 %. Further it is obviously more dangerous to choose too thin an absorber than one that is too thick, as the curve rises asymptotically towards infinity for $\mu d \to 0$. This must especially be remembered in connection with Parratt's thickness effect [4], which calls for the thinnest possible absorbers.

When an absorption edge is recorded with the aim of obtaining good statistics over the whole curve, d can be chosen somewhere in the interval

$$\left(d_1 = \frac{x_{\text{opt}}}{\mu_1};\; d_2 = \frac{x_{\text{opt}}}{\mu_2}\right),$$

where μ_1 and μ_2 are the linear absorption coefficients on either side of the edge. The intermediate thickness d_m is best chosen so that both $\mu_1 d_m$ and $\mu_2 d_m$ give rise to the same relative standard deviations. Often only part of the absorption spectrum is of special interest, e.g. the fine structure on the high energy side of an edge, in which case d_{opt} is obtained from the average μ in the region.

As pointed out above, the time distribution $\frac{t}{t_0}$ must be varied when μ varies. This is done according to Eq. (20) or (11 b).

Practical procedure

To determine x_{opt} and y_{opt} from the table, $\alpha = \frac{\nu_0}{n_0}$ and $\beta = \frac{\mu_\nu}{\mu}$ must be known. According to Eq. (13) μ must further be known in order to obtain d_{opt}. The parameters α, β, μ may be found in the following manner.

The intensity I_0 of the incident radiation is recorded as a function of the X-ray tube angle φ_1, while φ_2 is kept constant (see Fig. 4). Alternatively the positions of X-ray tube and detector can be kept constant while the crystal is turned. In the former case a curve like the upper one in Fig. 5 is obtained, where the peak corresponds to Bragg reflexion ($\varphi_1 = \varphi_2$). The point c in the figure then represents the background intensity ν_0, and the peak point a represents n_0. This gives us $\frac{c}{a} = \frac{\nu_0}{n_0} = \alpha$. The procedure is then repeated with an absorber in the path of the ray (the lower curve). The thickness of this absorber must be known but may be chosen arbitrarily. The point d represents ν, and μ_ν is calculated according to Eq. (4). Finally we get μ from the equation (2), point b corresponding to n. Now $\frac{\mu_\nu}{\mu} = \beta$ as well as d_{opt} and $\left(\frac{t}{t_0}\right)_{\text{opt}}$ can be calculated.

Besides giving the parameters necessary to find d_{opt} and $\left(\frac{t}{t_0}\right)_{\text{opt}}$, a curve like the one in Fig. 5 also determines ν_0 and ν for the actual calculation of μ once the optimum conditions are satisfied. As mentioned earlier, ν_0 and ν are independent of the wavelength setting of the spectrograph within not too wide a region. In fact, for a certain spectrograph ν_0 is mainly determined by the operating conditions of the X-ray tube, and as long as these are constant the background varies very little. Thus the determination of ν_0 and ν need not be repeated for each single absorption measurement but only made now and then as a check on undisturbed conditions.

The method clearly shows that the statistical error in the background determination can be made negligible by choosing long enough counting periods per point and/or by changing φ_1 in small steps. The assumption as to these errors made earlier in this paper is thus justified.

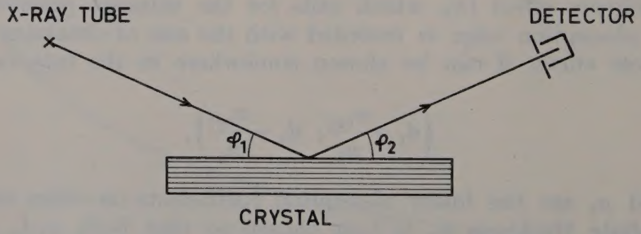

Fig. 4.

Finally it should be noted that there are other factors than the statistical accuracy which normally have to be considered when choosing an absorber thickness. The "thickness effect" described by Parratt [4] is one, and in some cases absorbers of a predetermined thickness may be impossible to manufacture.

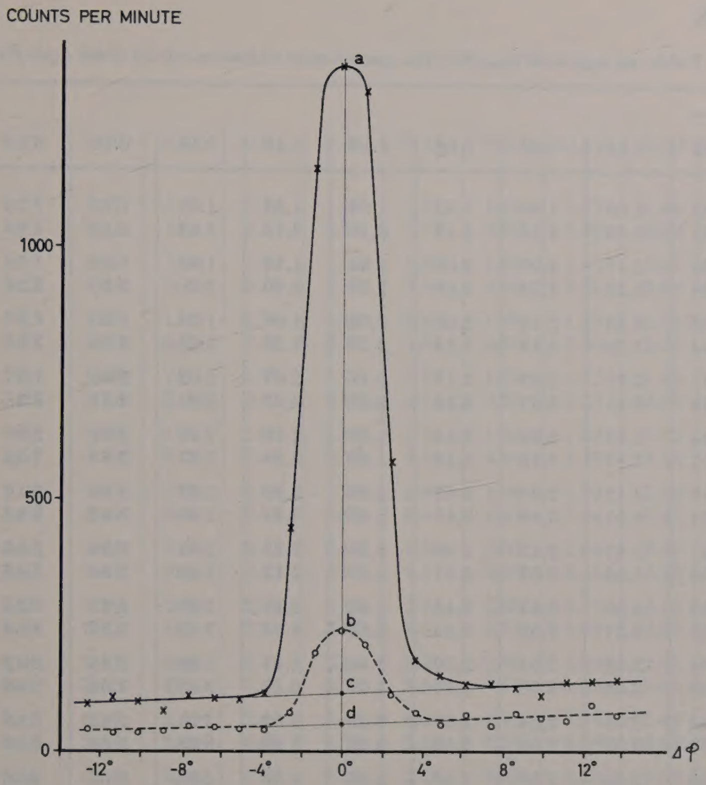

Fig. 5. The variation of the incident radiation I_0 when the glancing angle of the X-radiation incident on the crystal is varied. $\Delta\varphi$ is the deviation from the Bragg angle.

The present investigation is not aimed at prescribing a definite thickness in all cases, but mainly at giving advice about the limits tolerated from a statistical point of view.

ACKNOWLEDGEMENTS

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APPENDIX

Table of x_{opt} and y_{opt} for the parameter values $\alpha = 0.02$–0.60 and $\beta = 0.10$–1.90, β

	$\alpha \rightarrow$ 0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.
β ↓ 0.00	2.24 3.31	2.09 3.23	1.99 3.18	1.92 3.16	1.86 3.15	1.81 3.14	1.77 3.14	1.73 3.15	1.70 3.15	1.68 3.16	1. 3.
0.10	2.30 3.38	2.17 3.29	2.07 3.24	2.00 3.22	1.94 3.21	1.89 3.20	1.85 3.20	1.82 3.20	1.79 3.20	1.76 3.20	1. 3.
0.20	2.36 3.43	2.24 3.35	2.15 3.31	2.08 3.28	2.03 3.27	1.98 3.26	1.94 3.25	1.90 3.25	1.87 3.25	1.84 3.25	1. 3.
0.30	2.41 3.48	2.31 3.41	2.23 3.37	2.17 3.35	2.11 3.33	2.07 3.32	2.03 3.31	2.00 3.31	1.97 3.31	1.94 3.31	1. 3.
0.40	2.45 3.51	2.37 3.47	2.30 3.43	2.25 3.41	2.20 3.39	2.16 3.38	2.12 3.37	2.09 3.37	2.06 3.36	2.03 3.36	2. 3.
0.50	2.48 3.54	2.42 3.51	2.37 3.48	2.32 3.47	2.28 3.45	2.25 3.44	2.22 3.43	2.19 3.43	2.16 3.42	2.14 3.42	2. 3.
0.60	2.51 3.56	2.47 3.54	2.43 3.53	2.39 3.51	2.36 3.50	2.33 3.49	2.31 3.49	2.28 3.48	2.26 3.48	2.24 3.47	2. 3.
0.70	2.53 3.58	2.50 3.57	2.47 3.56	2.45 3.55	2.43 3.54	2.41 3.54	2.39 3.53	2.37 3.53	2.35 3.52	2.33 3.52	2. 3.
0.80	2.54 3.59	2.53 3.58	2.51 3.58	2.50 3.57	2.48 3.57	2.47 3.57	2.46 3.57	2.45 3.56	2.43 3.56	2.42 3.56	2. 3.
0.90	2.55 3.59	2.54 3.59	2.54 3.59	2.53 3.59	2.53 3.59	2.52 3.59	2.51 3.58	2.51 3.58	2.50 3.58	2.50 3.58	2. 3.
1.00	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2. 3.
1.10	2.56 3.59	2.57 3.59	2.57 3.59	2.57 3.59	2.58 3.59	2.58 3.59	2.59 3.59	2.59 3.59	2.60 3.58	2.60 3.58	2. 3.
1.20	2.56 3.59	2.57 3.59	2.58 3.58	2.58 3.58	2.59 3.58	2.60 3.58	2.61 3.57	2.61 3.57	2.62 3.57	2.63 3.57	2. 3.
1.30	2.56 3.58	2.57 3.58	2.58 3.58	2.59 3.57	2.60 3.57	2.61 3.56	2.62 3.56	2.63 3.55	2.64 3.55	2.65 3.54	2. 3.
1.40	2.57 3.58	2.57 3.58	2.58 3.57	2.59 3.56	2.60 3.55	2.61 3.55	2.62 3.54	2.63 3.53	2.64 3.52	2.65 3.51	2. 3.
1.50	2.57 3.58	2.57 3.57	2.58 3.56	2.59 3.55	2.60 3.54	2.61 3.53	2.62 3.52	2.63 3.51	2.65 3.50	2.66 3.49	2. 3.
1.60	2.57 3.58	2.57 3.56	2.58 3.55	2.59 3.54	2.60 3.53	2.61 3.51	2.62 3.50	2.63 3.49	2.64 3.47	2.66 3.46	2. 3.
1.70	2.57 3.58	2.57 3.56	2.58 3.54	2.59 3.53	2.60 3.51	2.61 3.50	2.62 3.48	2.63 3.47	2.64 3.45	2.65 3.43	2. 3.
1.80	2.56 3.57	2.57 3.56	2.58 3.54	2.59 3.52	2.60 3.50	2.61 3.49	2.62 3.47	2.63 3.45	2.64 3.43	2.65 3.41	2. 3.
1.90	2.56 3.57	2.57 3.55	2.58 3.53	2.59 3.51	2.60 3.49	2.61 3.47	2.61 3.45	2.62 3.43	2.63 3.41	2.64 3.40	2. 3.
∞	2.56 3.57	2.57 3.51	2.57 3.51	2.58 3.45	2.58 3.45	2.59 3.39	2.59 3.39	2.60 3.34	2.60 3.34	2.61 3.28	2. 3.

upper figure in each pair of optimum values is x_{opt} and the lower one y_{opt}.

4	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40	0.45	0.50	0.55	0.60
3 7	1.61 3.18	1.59 3.19	1.57 3.21	1.55 3.21	1.54 3.23	1.52 3.24	1.51 3.25	1.49 3.26	1.47 3.29	1.44 3.32	1.42 3.35	1.40 3.37
1 2	1.69 3.22	1.67 3.23	1.65 3.24	1.63 3.25	1.61 3.26	1.60 3.27	1.59 3.28	1.57 3.29	1.54 3.31	1.51 3.34	1.49 3.36	1.47 3.39
9 6	1.77 3.27	1.75 3.27	1.73 3.28	1.72 3.29	1.70 3.29	1.68 3.30	1.67 3.31	1.66 3.32	1.62 3.34	1.60 3.36	1.57 3.39	1.55 3.41
9 1	1.87 3.31	1.85 3.32	1.83 3.32	1.81 3.33	1.79 3.33	1.78 3.34	1.76 3.35	1.75 3.35	1.72 3.37	1.69 3.39	1.66 3.41	1.64 3.43
9 6	1.97 3.36	1.95 3.37	1.93 3.37	1.91 3.37	1.89 3.38	1.88 3.38	1.86 3.39	1.85 3.39	1.82 3.40	1.79 3.42	1.76 3.44	1.74 3.45
9 2	2.07 3.42	2.05 3.42	2.03 3.42	2.02 3.42	2.00 3.42	1.99 3.43	1.97 3.43	1.96 3.43	1.92 3.44	1.90 3.45	1.87 3.46	1.85 3.48
0 7	2.18 3.47	2.16 3.47	2.15 3.47	2.13 3.47	2.11 3.47	2.10 3.47	2.09 3.47	2.07 3.47	2.04 3.48	2.02 3.49	1.99 3.49	1.97 3.50
0 2	2.29 3.52	2.27 3.52	2.26 3.51	2.25 3.51	2.23 3.51	2.22 3.51	2.21 3.51	2.20 3.52	2.17 3.52	2.15 3.52	2.12 3.53	2.10 3.53
0 6	2.39 3.56	2.38 3.56	2.37 3.55	2.36 3.55	2.35 3.55	2.34 3.55	2.33 3.55	2.32 3.55	2.30 3.55	2.28 3.55	2.26 3.56	2.25 3.56
9 8	2.48 3.58	2.48 3.58	2.47 3.58	2.47 3.58	2.46 3.58	2.46 3.58	2.45 3.58	2.45 3.58	2.43 3.58	2.42 3.58	2.41 3.58	2.40 3.58
6 9	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59	2.56 3.59
1 8	2.61 3.58	2.62 3.58	2.62 3.58	2.63 3.58	2.63 3.58	2.64 3.58	2.64 3.58	2.65 3.58	2.66 3.58	2.67 3.58	2.69 3.58	2.70 3.58
4 6	2.65 3.56	2.66 3.56	2.67 3.56	2.68 3.55	2.69 3.55	2.70 3.55	2.71 3.55	2.72 3.55	2.74 3.54	2.77 3.54	2.80 3.54	2.83 3.54
7 3	2.68 3.53	2.69 3.52	2.70 3.52	2.71 3.51	2.72 3.51	2.73 3.50	2.75 3.50	2.76 3.50	2.79 3.49	2.83 3.48	2.87 3.47	2.91 3.46
8 0	2.69 3.49	2.70 3.48	2.71 3.47	2.73 3.47	2.74 3.46	2.75 3.45	2.77 3.44	2.78 3.44	2.82 3.42	2.86 3.40	2.91 3.38	2.97 3.37
8 6	2.69 3.45	2.70 3.44	2.72 3.43	2.73 3.42	2.75 3.41	2.76 3.40	2.77 3.39	2.79 3.37	2.83 3.35	2.88 3.32	2.93 3.29	2.99 3.26
8 3	2.69 3.42	2.70 3.40	2.72 3.39	2.73 3.37	2.74 3.36	2.76 3.34	2.77 3.33	2.79 3.31	2.83 3.28	2.88 3.24	2.93 3.20	2.99 3.15
7 0	2.69 3.38	2.70 3.37	2.71 3.35	2.72 3.33	2.74 3.31	2.75 3.29	2.77 3.28	2.78 3.26	2.82 3.21	2.87 3.16	2.92 3.11	2.98 3.06
7 7	2.68 3.35	2.69 3.33	2.71 3.31	2.72 3.29	2.73 3.27	2.74 3.25	2.76 3.23	2.77 3.21	2.81 3.15	2.86 3.10	2.91 3.04	2.96 2.97
6 5	2.67 3.33	2.69 3.31	2.70 3.28	2.71 3.26	2.72 3.24	2.73 3.21	2.75 3.19	2.76 3.17	2.80 3.10	2.84 3.04	2.89 2.97	2.94 2.90
2 2	2.63 3.18	2.63 3.18	2.64 3.12	2.65 3.08	2.65 3.08	2.66 3.03	2.67 2.98	2.68 2.94	2.70 2.86	2.72 2.78	2.75 2.67	2.79 2.53

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